Beneath the surface

KZ from non-pert DS

Nikita Nekrasov

Knizhnik - Zamolodchikov equation

In 2d CFT with current algebra symmetry

equation obeyed by current conformal blocks

$$\Psi(\bar{A}) = \left\langle e^{\int_{\Sigma} \mathrm{Tr}\, j \bar{A}} \right\rangle$$

$(0,1)$-form valued in $g^{\mathbb{C}}$

2d worldsheet

Kac-Moody affine algebras

dim $(1,0)$

$$j^a(z) j^b(w) \sim \frac{k\,\delta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c}{z-w} j^c(w) + \cdots$$

j_z

background field

(0,1)-form valued in $\mathfrak{g}^{\mathbb{C}}$

$$\Psi(\bar{A}) = \left\langle e^{\int_{\Sigma} \mathrm{Tr}\, j \bar{A}} \right\rangle$$

2d worldsheet

open dense subset

$$\mathcal{M}_G^{\text{flat}} = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

topological object

$$\Psi(g^{-1}\bar{A}g + g^{-1}\bar{\partial}g) = e^{k\left(S_{WZNW}(g) + \int \mathrm{Tr}\, \bar{A}\, g^{-1}\partial g\right)}\, \Psi(\bar{A})$$

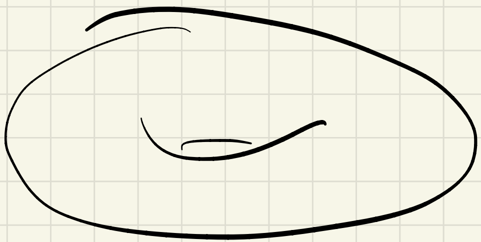
$g^{-1}\bar{A}g + g^{-1}\bar{\partial}g$ "=" $\bar{A}^{g}$

$$\Updownarrow$$

$g \mapsto g_1 g_2$

Ψ - holomorphic section of $\mathcal{L}^{\otimes k}$ over $\mathrm{Bun}_{G_{\mathbb{C}}}(\Sigma) = \{\bar{A} \sim \bar{A}^{g}\}$

can be descr'd in finite-dim terms $(g > 1)$

$\Sigma =$ 

$$\wr\wr$$

$$E_\tau = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}, \quad \operatorname{Im}\tau > 0$$

as complex curve

$$\Sigma_\tau$$

$$\operatorname{Bun}_{G^{\mathbb{C}}}(\Sigma)^{ss} = (E_\tau^{r})/W \simeq W\mathbb{CP}^{r}_{(a_0:a_1:\dots:a_r)}$$

$$\mathcal{M}_G^{flat}(\Sigma) = (T\times T)/W$$

does not depend on the complx str. τ

$$H^0\left(\mathrm{Bun}_{G_{\mathbb{C}}}(\Sigma)^{ss}, \mathcal{L}^{\otimes k}\right) =$$

geometric quantization of $\mathcal{M}_G^{flat}(\Sigma)$

operators $\longleftarrow$ $f(C) = \mathrm{Tr}_R\, P\exp\oint_C A$

identified for different choices of τ

$$\left\langle e^{\int \mathrm{Tr}\, j \bar{A} + \mu T} \right\rangle$$

(2,0) of stress tensor

Beltrami differential $\longleftrightarrow$ variation of cmplx structure on Σ

WZNW

$$T \sim \frac{1}{k+h^\vee} :\mathrm{Tr}\, j^2:$$

$$\left(\frac{\partial}{\partial \mu} - \frac{\partial^2}{\partial \bar{A}^2}\right)\Psi(\bar{A}, \mu) = 0$$

" heat equation "

+ Norton Lee

+ Oleksander Tsymbalyuk

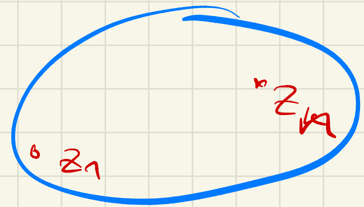
+ Saebyeok Jeong

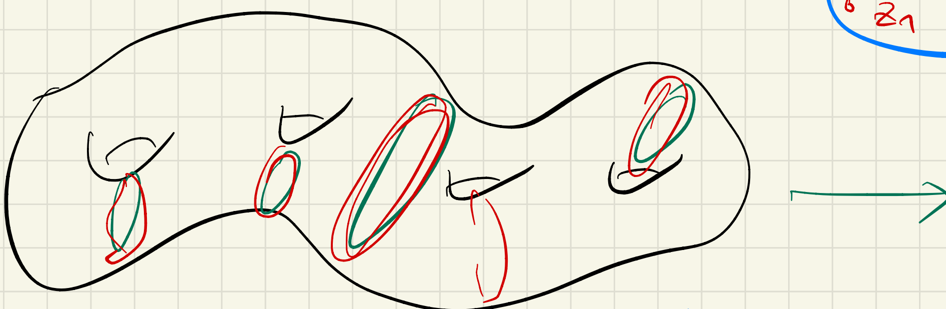
$\Psi(z_1,\dots,z_n) \in (R_1\otimes \dots \otimes R_k)^G$ $\qquad g=0$

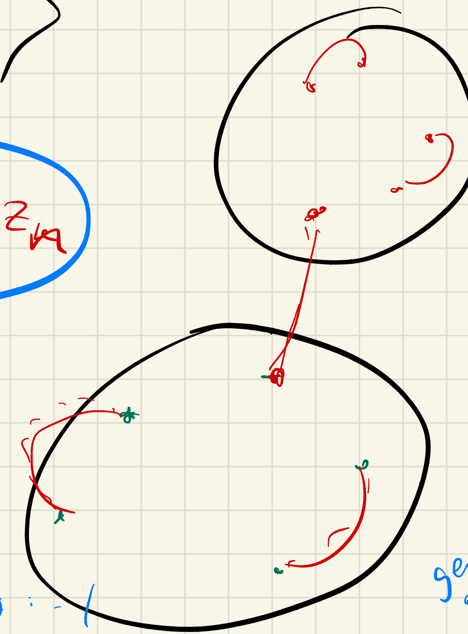
$" = " \langle V_1(z_1) \quad \dots \quad V_k(z_n) \rangle$

R_i

labelled by Reps of G
vectors







$(1\otimes \cdots \otimes \cdots 1)$

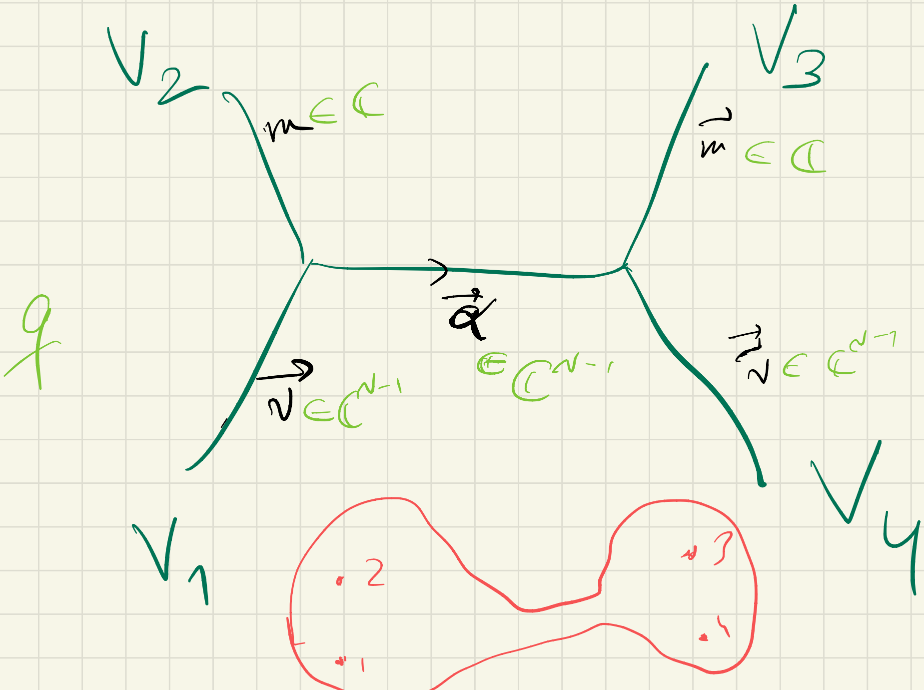
$$(k+h^\vee)\frac{d}{dz_i}\Psi - \sum_{j\neq i}\frac{T_i^a\otimes T_j^a}{z_i-z_j}\Psi = 0 \qquad T_i^a = T_{R_i}(t^a)$$

generator of g

$i=1,\dots,n$

KZ exist for complex k, spins, etc.

$$\mathfrak{g} = \mathfrak{sl}_N$$



$$\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$$

4pt block

$$q = \frac{z_2 - z_1}{z_3 - z_1} \cdot \frac{z_3 - z_4}{z_2 - z_4}$$

cross-ratio

(complx structure)

V_1 Verma module of lowest weight

V_2 HW built on $W \cong \mathbb{C}^N$

V_3 HW —— " —— $W^* \cong (\mathbb{C}^N)^*$

V_4 Verma module of highest weight

$$[J^a{}_b, J^{a'}{}_{b'}] = \delta^{a'}{}_b J^a{}_{b'} - \delta^a{}_{b'} J^{a'}{}_b$$

V_1

$J^a{}_b \Omega_{\vec{\nu}} = 0 \qquad a < b$

$\nu_a \in \mathbb{C}$

$J^a{}_a \Omega_{\vec{\nu}} = \nu_a \Omega_{\vec{\nu}} \qquad a = 1, \dots, N$

$V_1 = \mathbb{C}[J^a{}_b, a > b] \Omega_{\vec{\nu}}$

V_y

$J^a{}_b \, \tilde{\Omega}_{\vec{\gamma}} = 0 \qquad a > b$

$\tilde{\gamma}_a \in \mathbb{C}$

$J^a{}_a \, \tilde{\Omega}_{\vec{\gamma}} = \tilde{\gamma}_a \, \tilde{\Omega}_{\vec{\gamma}} \qquad a = 1, \ldots, N$

$V_\gamma = \mathbb{C}\left[J^a{}_b \, , \, a < b \right] \tilde{\Omega}_{\vec{\gamma}}$

HW modules

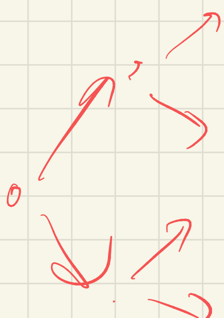
$$W \cong \mathbb{C}^N \ni z = \sum_{a=1}^{N} z^a e_a$$

$\mu_c \in \mathbb{C}$

$c = 1, \dots, N$

$$J^a{}_b = \prod_{c=1}^{N} z_c^{-\mu_c} \left(-z^a \frac{\partial}{\partial z^b} \right) \prod_{c=1}^{N} z_c^{\mu_c}$$

acting on $V_2 = \mathbb{C}[z_a, z_a^{-1}]^{\deg 0}$



$$J^a{}_b \cdot 1 = -\mu_b \, z^a (z^b)^{-1}$$

$C_1 = J^a{}_a$ $\qquad$ $C_2 = J^a{}_b J^b{}_a$ depend only on $m = \sum_c \mu_c$

$W^* \cong (\mathbb{C}^N)^* \ni \tilde z = \sum_{a=1}^{N} \tilde z_a \tilde e^a$

add'l twist

$$J^a{}_b = \prod_{c=1}^{N} \tilde z_c^{-\tilde\mu_c} \left(\tilde z_b \frac{\partial}{\partial \tilde z_a} \right) \prod_{c=1}^{N} \tilde z_c^{\tilde\mu_c}$$

$\tilde\mu_c \in \mathbb{C}$

$c=1,\dots,N$

acting on $V_z = \mathbb{C}[\tilde z_a, \tilde z_a^{-1}]^{\deg 0}$

$\mathrm{Sym}^\ell W^*$

$J^a{}_b \cdot 1 = \tilde\mu_a \tilde z^b (\tilde z^a)^{-1}$

Casimirs only depend on $\tilde m = \sum_c \tilde\mu_c$

$$\mathcal{H} = \left(\dot{V}_1 \otimes \ddot{V}_2 \check{\otimes} \dot{V}_3 \otimes \ddot{V}_4 \right)$$

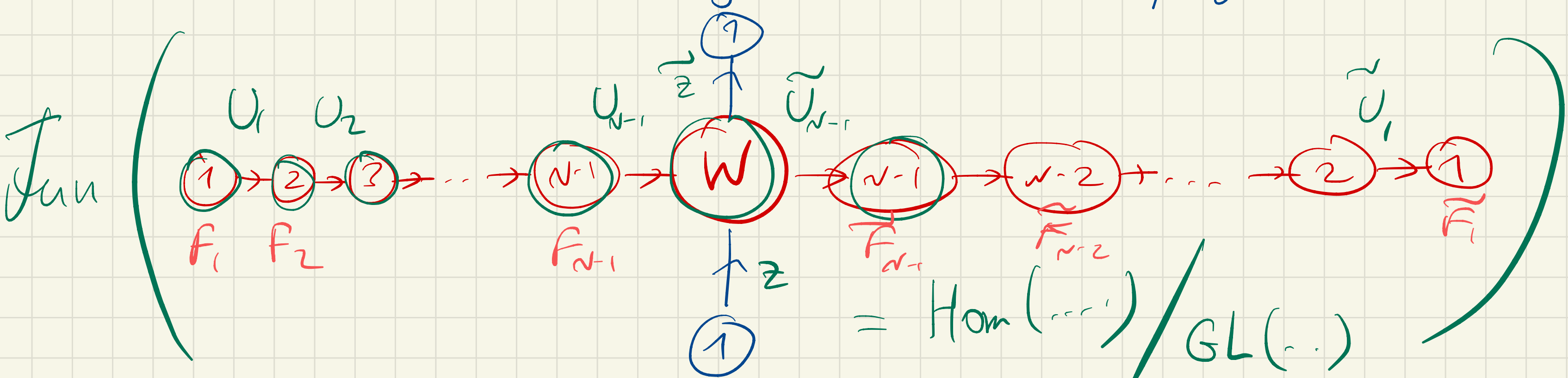
$\mathfrak{sl}_N$

$W \cong \mathbb{C}^N$

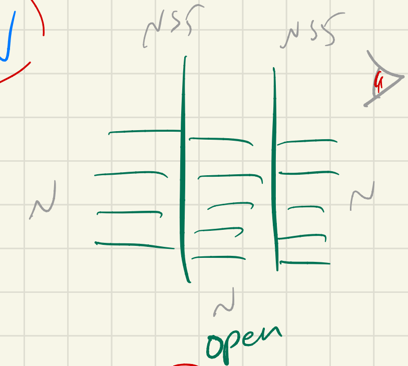
"Fun $F(W)$"   "Funn$(W)^\circ$"   "Fun$(W^*)^\circ$"   Fun $F(W^*)$

quiver description!

Verma's have something to do with flag varieties



$$\Psi \in \mathcal{H} = \left(V_1 \otimes V_2 \otimes V_3 \otimes V_4 \right)^{\mathfrak{sl}_N}$$



$$\Vert$$

$$H^0\left(\left(\underbrace{\mathbb{F}(W)}_{N-1} \times \underbrace{\mathbb{P}(W)}_{1} \times \underbrace{\mathbb{P}(W^*)}_{1} \times \underbrace{\mathbb{F}(W^*)}_{N-1} \right) / SL(W),\ \mathcal{L} \right)$$

Complete flags

$$W_1 \subset W_2 \subset \cdots \subset W_{N-1} \subset W$$

Complete flags

$$W^* \supset \tilde{W}_{N-1} \supset \tilde{W}_{N-2} \supset \cdots \supset \tilde{W}_1$$

$$\hat{\mathbb{L}} = \bigotimes_{i=1}^{N-1} \mathcal{L}_i^{\nu_i} \otimes \tilde{\mathcal{L}}_i^{\tilde{\nu}_i} \otimes L^m \otimes \tilde{L}^{\tilde{m}}$$

2N param's

$$\mathcal{L}_i = \det W_i$$

$$\tilde{\mathcal{L}}_i = \det \tilde{W}_i$$

$$L = \underset{\mathbb{P}(W)}{\mathcal{O}(1)}$$

$$\tilde{L} = \underset{\mathbb{P}(W^*)}{\mathcal{O}(1)}$$

$U_i : \mathbb{C}^i \to \mathbb{C}^{i+1}$

$U_i \mapsto g_{i+1} U_i g_i^{-1}$

$\tilde{U}_i : \mathbb{C}^{i+1} \longrightarrow \mathbb{C}^i$

$\tilde{U}_i \mapsto \tilde{g}_i \tilde{U}_i \tilde{g}_{i+1}^{-1}$

$z : \mathbb{C}^1 \to W$

Invariants?

$\tilde{z} : W \to \mathbb{C}^1$

linear operator : $F_i \to W$

$\Pi_i = \underline{\Lambda^i (\overbrace{U_{N-1} U_{N-2} \cdots U_i}) : \Lambda^i F_i} \longrightarrow \Lambda^i W$

$\Lambda^i F_i \simeq \mathbb{C}$, $\Lambda^i W \simeq \binom{N}{i}$

$g \in GL(N)$

$$V_i = \frac{\tilde{Z} \wedge \tilde{\Pi}^{i-1}(\Pi_i) \cdot \tilde{\Pi}^{i}(z \wedge \Pi_{i-1})}{\tilde{Z}(z) \cdot \tilde{\Pi}^{i}(\Pi_i) \cdot \tilde{\Pi}^{i-1}(\Pi_{i-1})}$$

invariant

$GL(W)$

$i = 1, \dots, N-1$

$SL(N)$ analogues of the cross-ratio

$N = 2$

$$\Big(\underbrace{F(W) \times \mathbb{P}(W) \times \mathbb{P}(W^*) \times F(W^*)}_{\text{compact}}\Big)\Big/SL(W) \simeq \left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right)\Big/SL(2)$$

topology

$\in \left(V_1 \otimes V_2 \otimes V_3 \otimes V_4 \right)^{\mathfrak{gl}_N}$

$J^a_b = \sum_{m=1}^{N-1} U^a_{N-1|m} \frac{\partial}{\partial U^b_{N-1|m}}$

$\Psi[u_i, \tilde{u}_i, z, \tilde{z}] =$

V_1 3N-1 param.

$= \prod_{i=1}^{N-1} \left((\tilde{z} \wedge \tilde{n}^{i-1})(n_i) \right)^{\nu_i} \left(\tilde{n}^i (z \wedge n_{i-1}) \right)^{\tilde{\gamma}_i} \left(\tilde{n}^i (n_i) \right)^{\gamma_i}$

$\cdot \; \cancel{\mathcal{X}}(\nu_1, \ldots, \nu_{N-1}; q)$

2nd diff. operators in ν_i

$\left(K z \right.$ for 4 pts | depend on 2N param.

$\boxed{\varkappa \frac{d}{dq} \Psi = \left(\frac{\hat{H}_0}{q} + \frac{\hat{H}_1}{q-1} \right) \Psi}$ $\leftarrow$ non-pert. Dyson Sch.

q is not explicit

$$\left(\mathbb{F}(W) \times \mathbb{P}(W) \times \mathbb{P}(W^*) \times \mathbb{F}(W^*) \right) /\!/_{SL(W)} = \mathcal{Z}$$

$$\subset \quad \mathrm{Hitchin}\left(\mathbb{P}^1, \{0, q, 1, \infty\} \right)$$

$$\left(\mathcal{O}_0^{\mathbb{C}} \times \mathcal{O}_q^{\mathbb{C}} \times \mathcal{O}_1^{\mathbb{C}} \times \mathcal{O}_\infty^{\mathbb{C}} \right) /\!/_{SL(N)}$$

$\phi_0 = \mathrm{res}_0 \phi$ = semisimple

ϕ_∞ semisimple

ϕ_q — minimal (scalar - rank 1)

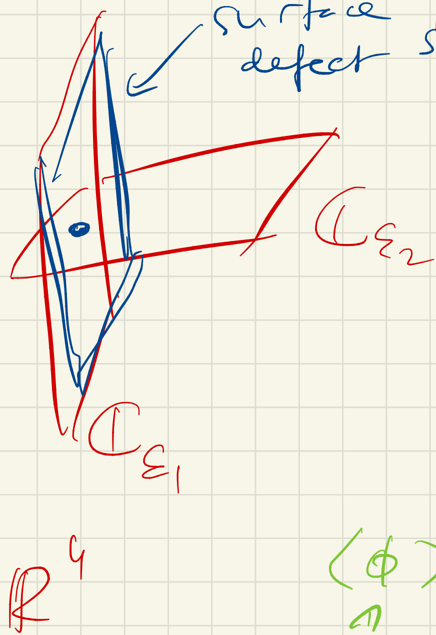
ϕ_1

$$\mathcal{O} \rightleftarrows \mathcal{O} \rightleftarrows \mathcal{O} \cdots -$$

Start in 4d now

$\mathcal{N}=2$ $d=4$ $SU(N)$ gauge th. with $2N$ fund. hypers

(asymp. f. conf)



$SU(N) \to U(1)^{N-1}$

$SU(N)/U(1)^{N-1}$

$F(\omega) \simeq F(\omega^+)$ Ω-deformation

$\varepsilon_1, \varepsilon_2$

$m_1^+, \ldots, m_N^+$

$m_1^-, \ldots, m_N^-$

Masses of hypers

$q = \exp\left(i\vartheta - \frac{8\pi^2}{g^2}\right)$

$\langle \Phi \rangle \sim \mathrm{diag}(a_1, \ldots, a_N)$

vector multiplet

Coulomb moduli

$$\vec{v} \sim \left(\frac{m_i^+}{\varepsilon_1} - \frac{1}{N\varepsilon_1} \sum m^+ \right)$$

$$m = \frac{1}{\varepsilon_1} \sum m^+$$

$$\tilde{m} = \frac{1}{\varepsilon_1} \sum m^-$$

$$\vec{v} = \left(\frac{m_i^-}{\varepsilon_1} - \frac{1}{N\varepsilon_1} \sum m^- \right)$$

Represent. content

$\mathcal{H}$

$$x = \frac{\varepsilon_2}{\varepsilon_1}$$

$$(q, \quad \vec{\alpha} = \left(\frac{a_i}{\varepsilon_1} \right)_{i=1}^{N-1}$$

$$a_N = -\sum_{i=1}^{N-1} a_i$$

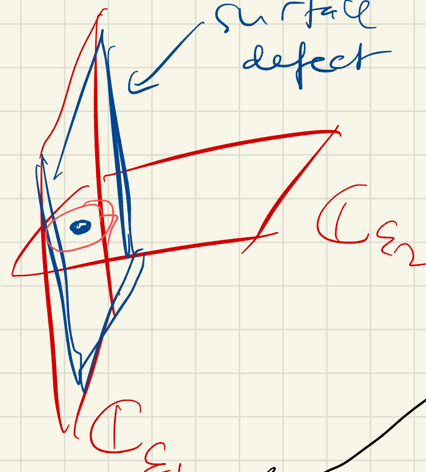
Supports σ-model valued in $F(W)$

$\Rightarrow$ Kähler moduli

$H^2(F(W), \mathbb{Z}) = \mathbb{Z}^{N-1}$

finite # of choices

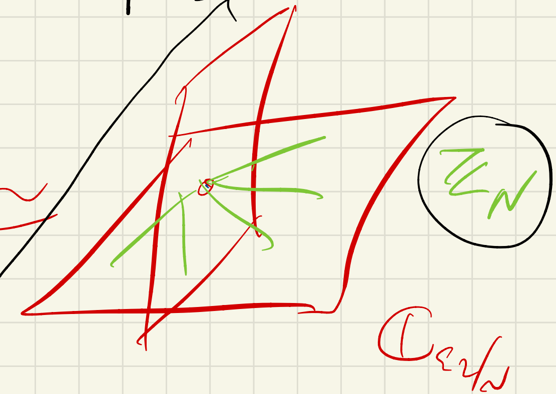
(ordering of m's
a's)



$q \to (q_0, \dots, q_{N-1})$

$(v_1, \dots, v_{N-1})$

$\simeq$



$\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$

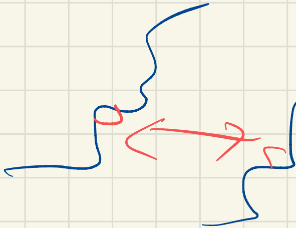
$\|$

$\mathbb{C} \times \mathbb{C}$

$(z_1, z_2) \mapsto (z_1, z_2 \omega)$

$\omega^N = 1$

On the 4d (+ 2d σ-model) side we are doing instanton enumeration



$$\Psi = \sum_{\vec{k} \in \mathbb{Z}_+^N} \vec{q}^{\,\vec{k}} \int_{\mathcal{M}_{\vec{k}}} \mathrm{Euler}(\ldots)$$

enumerative
discrete
topology

$\vec{q} = (q, \vec{v})$

complex geometry

$\vec{a}, \vec{m}, \varepsilon$ – equiv. parameters

$\simeq$

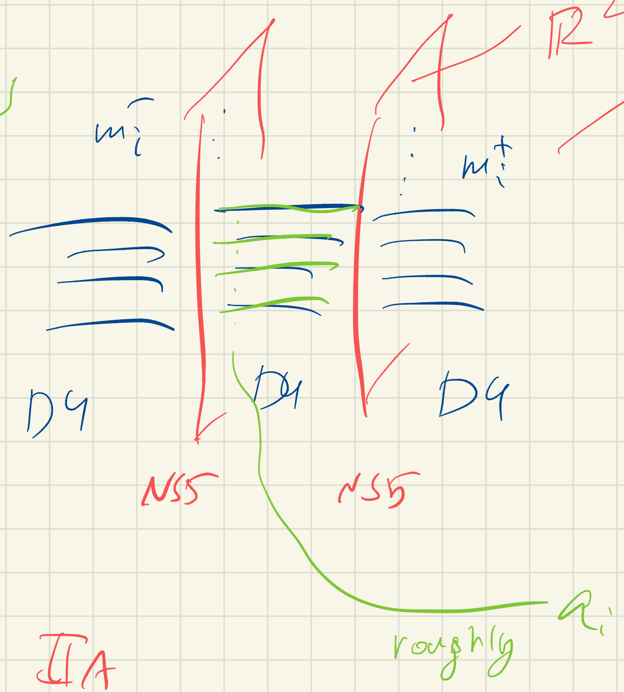
(Component of Affine flag variety)

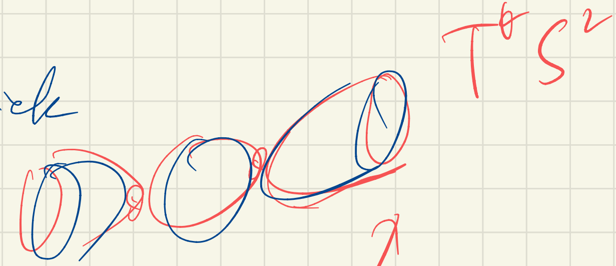
moduli space of (quasi)maps of $\mathbb{CP}^1 = (\mathbb{C}_{\varepsilon_1} \cup \{\infty\}) \Longrightarrow$ Flags

Hint

Kapustin's TST trick

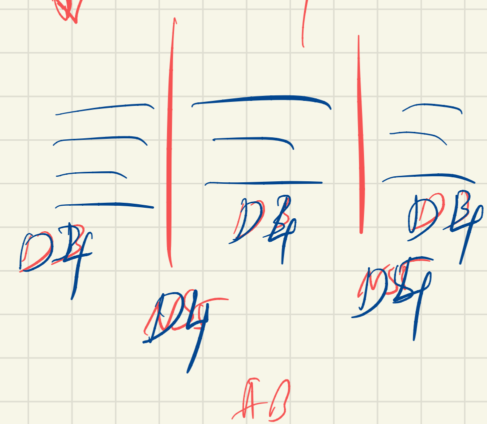
Witten's





Compactify on S^1 common to NS5's and D4's

TST

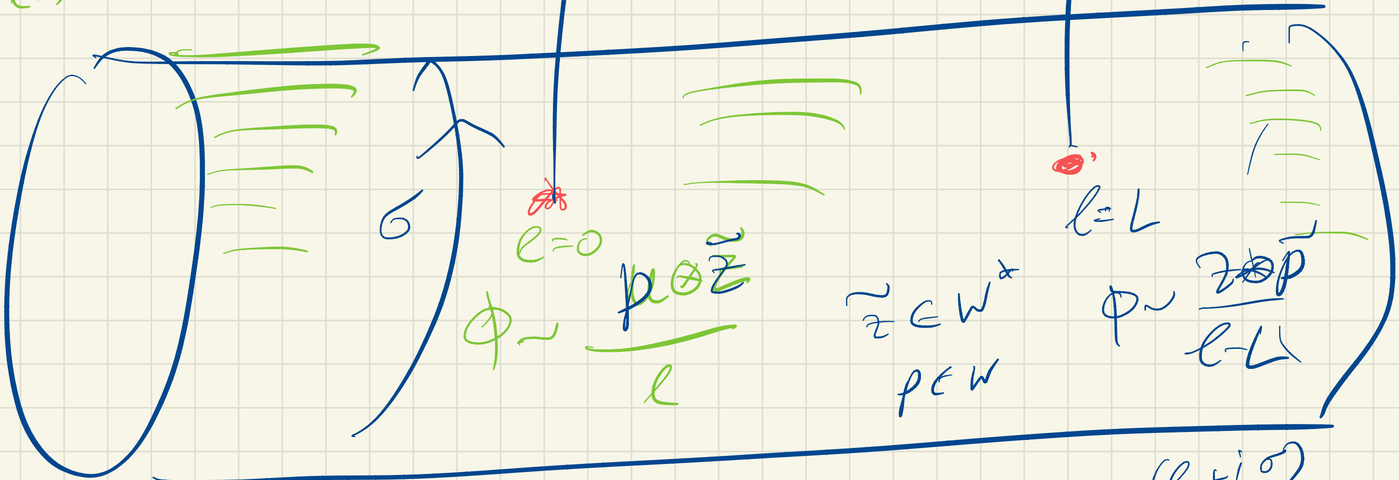


HW setup

$\Phi \sim \mathrm{diag}(m_i^-)$ $\quad \ell \to -\infty$

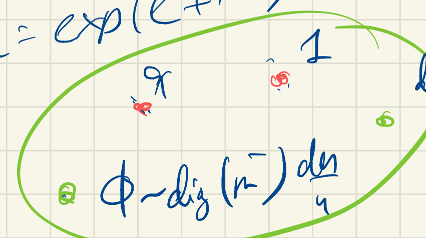
D^4 $\qquad\qquad\qquad$ D^4

$\Phi \sim \mathrm{diag}(m_i^+)$ $\quad \ell \to +\infty$



$\Phi \sim \dfrac{p \otimes \tilde{z}}{\ell}$ $\qquad \tilde{z} \in W^*$, $\; p \in W$ $\qquad \Phi \sim \dfrac{\tilde{z} \otimes p}{\ell - L}$

$\approx \quad u = \exp(\ell + i\sigma)$

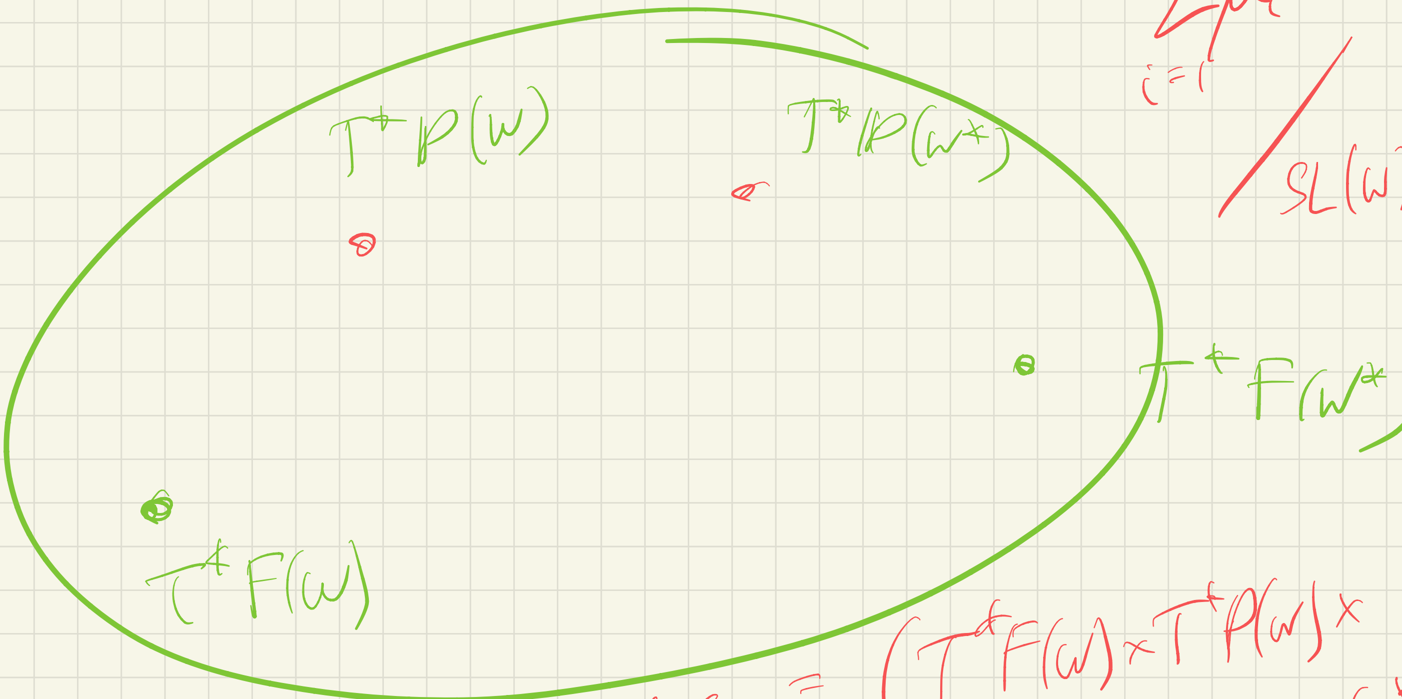


$\mathrm{diag}(m^+)\,\dfrac{du}{u}$

N D4 on $S^1 \times \mathbb{R}$

$\bar{\partial}_A \Phi = \sum \mu_i\, \delta^{(2)}(z - z_i)$ $\;\Big|\;$ rk N $\quad \Phi$ Higgs field

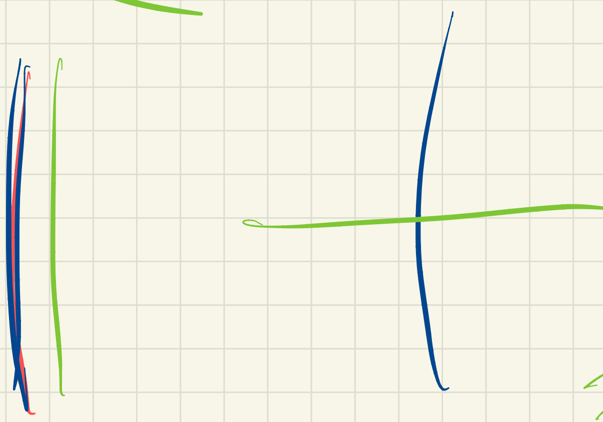
$$\sum_{i=1}^{4} \mu_i = 0 \Big/\!\!\Big/ SL(W)$$



Moduli space of vacua $= \left(T^*F(W) \times T^*P(W) \times T^*P(W^*) \times T^*F(W^*)\right) /\!\!/ SL(W)$

function

Q − observables $\longrightarrow$ $\left(V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes W \right)^{sl_N}$



"light" surface defects

≈ degenerate fields

WZW−wise